

Characterizations of Jordan $*$ -derivations on Banach $*$ -algebras

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Abstract: Suppose that \mathcal{A} is a real or complex unital Banach $*$ -algebra, \mathcal{M} is a unital Banach \mathcal{A} -bimodule, and $G \in \mathcal{A}$ is a left separating point of \mathcal{M} . In this paper, we investigate whether the additive mapping $\delta : \mathcal{A} \rightarrow \mathcal{M}$ satisfies the condition $A, B \in \mathcal{A}, AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G)$ characterize Jordan $*$ -derivations. Initially, we prove that if \mathcal{A} is a real unital C^* -algebra and $G = I$ is the unit element in \mathcal{A} , then δ (non-necessarily continuous) is a Jordan $*$ -derivation. In addition, we prove that if \mathcal{A} is a real unital C^* -algebra and δ is continuous, then δ is a Jordan $*$ -derivation. Finally, we show that if \mathcal{A} is a complex factor von Neumann algebra and δ is linear, then δ (non-necessarily continuous) is equal to zero.

Keywords: Jordan $*$ -derivation, Left Separating Point, C^* -algebra, Factor

1. Introduction

Let \mathcal{R} be an associative ring. By an *involution* on \mathcal{R} , we mean a mapping $*$ from \mathcal{R} into itself, such that $(AB)^* = B^*A^*$ and $(A^*)^* = A$ for each A, B in \mathcal{R} . A ring equipped with an involution is called a $*$ -ring. In [1], M. Brešar and J. Vukman give the concept of Jordan $*$ -derivations. An additive mapping δ from \mathcal{R} into itself is called a *Jordan $*$ -derivation* if

$$\delta(A^2) = \delta(A)A^* + A\delta(A) \tag{1}$$

for every A in \mathcal{R} . If \mathcal{A} is a $*$ -algebra and \mathcal{M} is an \mathcal{A} -bimodule, then it is clear that an additive mapping δ from \mathcal{A} into \mathcal{M} is a Jordan $*$ -derivation if and only if

$$\delta(AB) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) \tag{2}$$

for each A, B in \mathcal{A} .

The study of Jordan $*$ -derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones. It turns out that the question of whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of Jordan $*$ -derivations. For the results concerning this problem we refer to [5, 6, 12-15]

In [1], the authors study some algebraic properties of Jordan

$*$ -derivations. As a special case of [1, Theorem 1], we know that every Jordan $*$ -derivation δ from a complex unital $*$ -algebra \mathcal{A} into itself is of the form $\delta(A) = TA^* - AT$ for some T in \mathcal{A} . For non-unital $*$ -algebras, in [2], M. Brešar and B. Zalar prove that every Jordan $*$ -derivation δ from an algebra of all compact linear operators on a complex Hilbert space \mathcal{H} into itself is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$. But it is also an open question whether above result in [2] remains true in the real case.

Roughly speaking, it is much more difficult to study Jordan $*$ -derivations on real algebras than on complex algebras.

Nevertheless, in [10], P. Šemrl proves that every Jordan $*$ -derivation on $B(\mathcal{H})$ is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$, where \mathcal{H} is a real Hilbert space \mathcal{H} with $\dim \mathcal{H} > 1$, and in [2], the authors give a new proof of this result. In [11], P. Šemrl shows that every Jordan $*$ -derivation from a standard operator algebra \mathcal{A} on \mathcal{H} into $B(\mathcal{H})$ is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$, where \mathcal{H} is a real or complex Hilbert space \mathcal{H} with $\dim \mathcal{H} > 1$. In [9], X. Zhao and X. Qi prove that, under some mild conditions, an additive mapping δ from a $*$ -ring with a symmetric idempotent P into itself satisfies the condition

$$AB = P \Rightarrow \delta(AB) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) \tag{3}$$

if and only if δ is an additive Jordan $*$ -derivation.

2. The Main Results

It is not hard to see that when \mathcal{A} is a complex unital Banach $*$ -algebra, every linear Jordan $*$ -derivation from \mathcal{A} into a Banach \mathcal{A} -bimodule is zero. So, the notion of Jordan $*$ -derivation studied in this note is more appropriate in the setting of real Banach $*$ -algebras, and specially for real operator algebras and real C^* -algebras as illustrated by results in [10, 11].

Theorem 2.1. Suppose that \mathcal{A} is a real unital C^* -algebra and \mathcal{M} is a unital Banach \mathcal{A} -bimodule. If δ is an additive mapping

$$\begin{aligned} \delta(B) &= -B\delta(B^{-1})B^* = -B\delta(B^{-1}(I - B)^2 - B)B^* \\ &= -B\delta(B^{-1}(I - B)^2)B^* + B\delta(B)B^* \\ &= (I - B)^2\delta((I - B)^{-2}B)(I - B^*)^2 + B\delta(B)B^* \\ &= (I - B)^2\delta((I - B)^{-2} - (I - B)^{-1})(I - B^*)^2 + B\delta(B)B^* \\ &= (I - B)^2\delta((I - B)^{-2})(I - B^*)^2 - (I - B)^2\delta((I - B)^{-1})(I - B^*)^2 + B\delta(B)B^* \\ &= -\delta((I - B)^2) + (I - B)\delta(I - B)(I - B^*) + B\delta(B)B^* \\ &= -\delta(B^2) + B\delta(B) + \delta(B)B^* + \delta(B). \end{aligned} \tag{6}$$

It implies that $\delta(B^2) = B\delta(B) + \delta(B)B^*$, by $\delta(I) = 0$, we have that

$$\delta(T^2) = T\delta(T) + \delta(T)T^* \tag{7}$$

for every T in \mathcal{A} . it follows that δ is a Jordan $*$ -derivation.

Recall the definition of the left separating point. For an algebra \mathcal{A} and an \mathcal{A} -bimodule \mathcal{M} , G in \mathcal{A} is called a left separating point of \mathcal{M} if $GM = 0$ implies $M = 0$ for every M in \mathcal{M} .

Theorem 2.2. Suppose that \mathcal{A} is a real unital C^* -algebra, \mathcal{M} is a unital Banach \mathcal{A} -bimodule and G is a left separating point of \mathcal{M} . If δ is a continuous linear mapping from \mathcal{A} into \mathcal{M} that satisfies the condition

$$AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G), \tag{8}$$

By $G(I - tA)(I - tA)^{-1} = G$, it follows that

$$(G - tGA)\delta((I - tA)^{-1}) + \delta(G - tGA)(I - tA^*)^{-1} = \delta(G). \tag{10}$$

By (9) we can obtain that

$$(G - tGA)\delta\left(\sum_{n=0}^{\infty} t^n A^n\right) + \delta(G - tGA)\sum_{n=0}^{\infty} t^n (A^*)^n = \delta(G). \tag{11}$$

Since δ is a continuous linear mapping, we have that

$$\begin{aligned} \delta(G) &= \sum_{n=0}^{\infty} t^n G\delta(A^n) - \sum_{n=0}^{\infty} t^{n+1} GA\delta(A^n) + \sum_{n=0}^{\infty} t^n \delta(G)(A^*)^n - \sum_{n=0}^{\infty} t^{n+1} \delta(GA)(A^*)^n \\ &= \sum_{n=1}^{\infty} t^n [G\delta(A^n) - GA\delta(A^{n-1}) + \delta(G)(A^*)^n - \delta(GA)(A^*)^{n-1}] + G\delta(I) + \delta(G). \end{aligned} \tag{12}$$

(non-necessarily continuous) from \mathcal{A} into \mathcal{M} that satisfies the condition

$$AB = I \Rightarrow A\delta(B) + \delta(A)B^* = \delta(I), \tag{4}$$

then δ is a Jordan $*$ -derivation.

proof. It is easy to show that $\delta(I) = 0$. Let A be an invertible element in \mathcal{A} . By $AA^{-1} = I$, we can obtain that

$$A\delta(A^{-1}) + \delta(A)(A^*)^{-1} = 0. \tag{5}$$

For any T in \mathcal{A} , let n be a positive integer with $n > \|T\| + 1$ and $B = nI + T$. We have that B and $I - B$ are invertible in \mathcal{A} . Thus we know that

then δ is a Jordan $*$ -derivation.

Proof. Since $GI = G$, it follows that $G\delta(I) + \delta(G) = \delta(G)$. By the definition of the left separating point, we know that $\delta(I) = 0$.

Let A be a non-zero element in \mathcal{A} , it is well known that $I - tA$ is invertible in \mathcal{A} for every t in \mathbb{R} with $|t| < (\|A\|)^{-1}$, and we have that

$$(I - tA)^{-1} = \sum_{n=0}^{\infty} (tA)^n = \sum_{n=0}^{\infty} t^n A^n. \tag{9}$$

Since δ is a continuous linear mapping, it is easy to prove that δ is real linear. Thus $\delta(t^n B) = t^n \delta(B)$ for every B in \mathcal{A} and every positive integer n .

By $\delta(I) = 0$, it implies that

$$\sum_{n=1}^{\infty} t^n [G\delta(A^n) - GA\delta(A^{n-1}) + \delta(G)(A^*)^n - \delta(GA)(A^*)^{n-1}] = 0 \quad (13)$$

for every t in \mathbb{R} with $|t| < (\|A\|)^{-1}$. Consequently,

$$G\delta(A^n) - GA\delta(A^{n-1}) + \delta(G)(A^*)^n - \delta(GA)(A^*)^{n-1} = 0 \quad (14)$$

for all $n = 1, 2, \dots$. In particular, choose $n = 1$ and $n = 2$ in (14), respectively, we have the following two identities:

$$G\delta(A) - GA\delta(I) + \delta(G)A^* - \delta(GA) = 0 \quad (15)$$

and

$$G\delta(A^2) - GA\delta(A) + \delta(G)(A^*)^2 - \delta(GA)A^* = 0. \quad (16)$$

Multiplying A^* from the right side of (15) and by $\delta(I) = 0$, we obtain

$$G\delta(A)A^* + \delta(G)(A^*)^2 - \delta(GA)A^* = 0. \quad (17)$$

Comparing (16) and (17), we have that

$$G\delta(A^2) = GA\delta(A) + G\delta(A)A^*. \quad (18)$$

By the definition of the left separating point, we know that

$$\delta(A^2) = A\delta(A) + \delta(A)A^* \quad (19)$$

for every A in \mathcal{A} . Thus δ is a Jordan *-derivation.

We note that if \mathcal{A} is a complex unital Banach *-algebra and δ is a linear Jordan *-derivation from \mathcal{A} into its Banach bimodule \mathcal{M} , then δ is equal to zero. In fact, if

$$\delta(AB + BA) = A\delta(B) + \delta(B)A^* + B\delta(A) + \delta(A)B^* \quad (20)$$

for each A, B in \mathcal{A} , then by $\delta(I) = 0$, we have that

$$\delta(A(iI) + (iI)A) = (iI)\delta(A) + \delta(A)(-iI). \quad (21)$$

Hence $\delta(A) = 0$ for every A in \mathcal{A} . Thus by Theorems 2.1 and 2.2, we have the following two corollaries.

Corollary 2.1. Suppose that \mathcal{A} is a complex unital C^* -algebra and \mathcal{M} is a unital Banach \mathcal{A} -bimodule. If δ is a linear mapping (non-necessarily continuous) from \mathcal{A} into \mathcal{M} that

It follows that

$$(G - tGP)\delta(I - \frac{t}{t-1}P) + \delta(G - tGP)(I - \frac{t}{t-1}P) = \delta(G). \quad (27)$$

Multiplying $(t-1)$ from the left and right sides of (27), we have that

$$(G - tGP)\delta((t-1)I - tP) + \delta(G - tGP)((t-1)I - tP) = (t-1)\delta(G). \quad (28)$$

Hence, for any $t \neq 0, 1$, we can obtain that

$$t(GP\delta(P) - \delta(GP) + \delta(GP)P) + (\delta(GP) - G\delta(P) - \delta(G)P) = 0. \quad (29)$$

satisfies the condition

$$AB = I \Rightarrow A\delta(B) + \delta(A)B^* = \delta(I), \quad (22)$$

then δ is equal to zero.

Corollary 2.2. Suppose that \mathcal{A} is a complex unital C^* -algebra, \mathcal{M} is a unital Banach \mathcal{A} -bimodule and G is a left separating point of \mathcal{M} . If δ is a continuous linear mapping from \mathcal{A} into \mathcal{M} that satisfies the condition

$$AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G), \quad (23)$$

then δ is equal to zero.

For complex von Neumann algebras, we have the following result.

Theorem 2.3. Suppose that \mathcal{A} is a complex factor von Neumann algebra, \mathcal{M} is a unital Banach \mathcal{A} -bimodule and G is a left separating point of \mathcal{M} . If δ is a linear mapping (non-necessarily continuous) from \mathcal{A} into \mathcal{M} that satisfies the condition

$$AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G), \quad (24)$$

then δ is equal zero.

Proof. Since $GI = G$, it follows that

$$G\delta(I) + \delta(G) = \delta(G). \quad (25)$$

By the definition of the left separating point, we know that $\delta(I) = 0$.

For every projection P in \mathcal{A} and t in \mathbb{R} with $t \neq 1$, it is easy to show that

$$G(I - tP)(I - \frac{t}{t-1}P) = G. \quad (26)$$

Hence

$$\delta(GP) = G\delta(P) + \delta(G)P = G\delta(P) + \delta(G)P^*. \quad (30)$$

Since \mathcal{A} is a complex factor von Neumann algebra and by [4, Theorem 3], we know that every element in \mathcal{A} can be written as a complex linear combination of projections in \mathcal{A} . Thus we have that

$$\delta(GA) = G\delta(A) + \delta(G)A^* \quad (31)$$

For any T in \mathcal{A} , let n be a positive integer with $n > \|T\| + 1$ and $B = nI + T$. We have that B and $I - B$ are invertible in \mathcal{A} . By (33), we know that

$$\begin{aligned} \delta(B) &= -B\delta(B^{-1})B^* = -B\delta(B^{-1}(I - B)^2 - B)B^* \\ &= -B\delta(B^{-1}(I - B)^2)B^* + B\delta(B)B^* \\ &= (I - B)^2\delta((I - B)^{-2}B)(I - B^*)^2 + B\delta(B)B^* \\ &= (I - B)^2\delta((I - B)^{-2} - (I - B)^{-1})(I - B^*)^2 + B\delta(B)B^* \\ &= (I - B)^2\delta((I - B)^{-2})(I - B^*)^2 - (I - B)^2\delta((I - B)^{-1})(I - B^*)^2 + B\delta(B)B^* \\ &= -\delta((I - B)^2) + (I - B)\delta(I - B)(I - B^*) + B\delta(B)B^* \\ &= -\delta(B^2) + B\delta(B) + \delta(B)B^* + \delta(B). \end{aligned} \quad (34)$$

It implies that $\delta(B^2) = B\delta(B) + \delta(B)B^*$ for every B in \mathcal{A} , by $\delta(I) = 0$, we have that

$$\delta(T^2) = T\delta(T) + \delta(T)T^* \quad (35)$$

for every T in \mathcal{A} . By the discussion preceding Corollary 2.1, we know that δ is equal to zero.

3. Conclusion

Suppose that \mathcal{A} is a real or complex unital Banach $*$ -algebra and \mathcal{M} is a unital Banach \mathcal{A} -bimodule. An element G in \mathcal{A} is called a *left separating point* of \mathcal{M} if $GM = 0$ implies $M = 0$ for every M in \mathcal{M} . It is easy to see that every left invertible element in \mathcal{A} is a left separating point of \mathcal{M} .

In Section 2, we let G be in \mathcal{A} that is a left separating point of \mathcal{M} , and characterize the additive mapping δ from \mathcal{A} into \mathcal{M} satisfies the following condition

$$A, B \in \mathcal{A}, AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G). \quad (36)$$

Initially, we prove that if \mathcal{A} is a real unital C^* -algebra and $G = I$ is the identity in \mathcal{A} , then δ (non-necessarily continuous) is a Jordan $*$ -derivation. In addition, we prove that if \mathcal{A} is a real unital C^* -algebra and δ is continuous linear, then δ is a Jordan $*$ -derivation. Finally, we show that if \mathcal{A} is a complex factor von Neumann algebra and δ is a linear mapping, then δ (non-necessarily continuous) is equal to zero.

Data Availability

The data set supporting the conclusions is included within this article.

for every A in \mathcal{A} .

Let A be an invertible element in \mathcal{A} . By $GAA^{-1} = G$, we can obtain that

$$GA\delta(A^{-1}) + \delta(GA)(A^*)^{-1} = \delta(G). \quad (32)$$

By (31) and (32), it implies that

$$A\delta(A^{-1}) + \delta(A)(A^*)^{-1} = 0. \quad (33)$$

Conflicts of Interest

The authors declare that they have no competing interests.

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